

A gap rigidity for proper holomorphic maps

from \mathbf{B}^{n+1} to \mathbf{B}^{3n-1}

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Abstract Let $\mathbf{B}^{n+1} \subset \mathbb{C}^{n+1}$ be the unit ball in a complex Euclidean space, and let $\Sigma^n = \partial\mathbf{B}^{n+1} = S^{2n+1}$. Let $f : \Sigma^n \hookrightarrow \Sigma^N$ be a local CR immersion. If $N - n < 2n - 1$, the asymptotic vectors of the second fundamental form of f at each point form a subspace of the holomorphic tangent space of Σ^n of codimension at most 1. We exploit the successive derivatives of this relation and show that a linearly full local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, $N \leq 3n - 2$, can only occur when $N = n$, $2n$, or $2n + 1$. Together with the recent classification of the rational proper holomorphic maps from \mathbf{B}^{n+1} to \mathbf{B}^{2n+2} by Hamada, this gives a classification of the rational proper holomorphic maps from \mathbf{B}^{n+1} to \mathbf{B}^{3n-1} for $n \geq 3$.

Key words: proper holomorphic map, unit ball, CR immersion, gap rigidity

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Introduction

The purpose of this paper is to prove the following gap rigidity for the proper holomorphic maps between the unit balls in complex Euclidean spaces. Let $z = (z^0, z^i)$, $1 \leq i \leq n$, be the coordinates of \mathbb{C}^{n+1} . Let $\mathbf{B}^{n+1} \subset \mathbb{C}^{n+1}$ denote the unit ball.

Theorem *Let $F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{3n-1}$ be a proper holomorphic map that is C^3 up to the boundary, $n \geq 3$. Then, up to automorphisms of the unit balls, F is equivalent to one of the following three polynomial maps.*

A. *Linear embedding $F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{n+1} \subset \mathbf{B}^{3n-1}$.*

B. *Whitney map $F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{2n+1} \subset \mathbf{B}^{3n-1}$ defined by*

$$F(z) = (z^i, z^i z^0, (z^0)^2).$$

C. *$F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{2n+2} \subset \mathbf{B}^{3n-1}$ is defined for some ϕ , $0 < \phi < \frac{\pi}{2}$, by*

$$F(z) = (z^i, \cos(\phi) z^0, \sin(\phi) z^i z^0, \sin(\phi) (z^0)^2).$$

Proper holomorphic map between the unit balls is a subject with a long and fruitful history that goes back to Poincare, and Alexander, [Fo][DA2] for general references. More recently, Huang and Ji showed that every rational proper holomorphic maps from \mathbf{B}^{n+1} to \mathbf{B}^{2n+1} , $n \geq 2$, is equivalent to either the linear embedding or the Whitney map [HJ]. Hamada showed that every rational proper holomorphic maps from \mathbf{B}^{n+1} to \mathbf{B}^{2n+2} , $n \geq 3$, is equivalent to either one of the maps **A**, **B**, or **C** [Ha]. The one parameter family of maps **C** was introduced by D'Angelo [DA1].

Let $\Sigma^n = \partial \mathbf{B}^{n+1} = S^{2n+1}$. Since a proper holomorphic map $F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{3n-1}$ that is C^3 up to the boundary is rational [HJX, Corollary 1.4], in order to prove **Theorem** it suffices to show that any local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ lies in $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$ for $n \geq 3$, thereby reducing the problem to the case treated by Hamada. In fact, our analysis shows that a linearly full local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, $N \leq 3n - 2$, can only occur when $N = n$, $2n$, or $2n + 1$.

For a local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$, the asymptotic vectors of the second fundamental form of f at each point form a subspace of the holomorphic tangent space of Σ^n . *Rank* of the second fundamental form at a point is defined as the codimension of the asymptotic subspace. When $N - n < 2n - 1$, an algebraic analysis of CR Gauß equation shows that the rank is at most 1 [Iw]. The idea is then to explore the consequence of the successive derivatives of this relation to obtain the desired gap theorem. In this regard, a computations suggests that this type of *gap phenomena* may persist for local CR immersions $f : \Sigma^n \rightarrow \Sigma^{\mu(n+1)-1}$ with second fundamental form of rank 1 for $\mu \leq n - 1$. We make a relevant remark at the end of section 2.

Huang introduced the notion of geometric rank of a CR map between spheres [Hu]. A short computation shows that if a CR immersion has second fundamental form of rank 1, then it has geometric rank 1. We suspect that geometric rank is bounded above by the rank of the second fundamental form at a generic point.

One of our initial motivation was whether the complete system for CR maps in [Han] is subject to any geometric interpretation.

1 CR submanifold

We set up the basic structure equations for CR submanifolds in spheres. For general reference in CR geometry, [ChM][DA2][EHZ][Fo].

Let $\mathbb{C}^{N+1,1}$ be the complex vector space with coordinates $z = (z^0, z^A, z^{N+1})$, $1 \leq A \leq N$, and a Hermitian scalar product

$$\langle z, \bar{z} \rangle = z^A \bar{z}^A + i(z^0 \bar{z}^{N+1} - z^{N+1} \bar{z}^0).$$

Let Σ^N be the set of equivalence classes up to scale of null vectors with respect to this product. Let $\text{SU}(N+1, 1)$ be the group of unimodular linear transformations that leave the form $\langle z, \bar{z} \rangle$ invariant. Then $\text{SU}(N+1, 1)$ acts transitively on Σ^N , and

$$p : \text{SU}(N+1, 1) \rightarrow \Sigma^N = \text{SU}(N+1, 1)/P$$

for an appropriate subgroup P [ChM].

Explicitly, consider an element $Z = (Z_0, Z_A, Z_{N+1}) \in \text{SU}(N+1, 1)$ as an ordered set of $(N+2)$ -column vectors in $\mathbb{C}^{N+1,1}$ such that $\det(Z) = 1$, and that

$$\langle Z_A, \bar{Z}_B \rangle = \delta_{AB}, \quad \langle Z_0, \bar{Z}_{N+1} \rangle = -\langle Z_{N+1}, \bar{Z}_0 \rangle = i, \quad (1)$$

while all other scalar products are zero. We define $p(Z) = [Z_0]$, where $[Z_0]$ is the equivalence class of null vectors represented by Z_0 . The left invariant Maurer-Cartan form π of $\text{SU}(N+1, 1)$ is defined by the equation

$$dZ = Z\pi,$$

which is in coordinates

$$d(Z_0, Z_A, Z_{N+1}) = (Z_0, Z_B, Z_{N+1}) \begin{pmatrix} \pi_0^0 & \pi_A^0 & \pi_{N+1}^0 \\ \pi_0^B & \pi_A^B & \pi_{N+1}^B \\ \pi_0^{N+1} & \pi_A^{N+1} & \pi_{N+1}^{N+1} \end{pmatrix}. \quad (2)$$

Coefficients of π are subject to the relations obtained from differentiating (1) which are

$$\begin{aligned}
\pi_0^0 + \bar{\pi}_{N+1}^{N+1} &= 0 \\
\pi_0^{N+1} &= \bar{\pi}_0^{N+1}, \quad \pi_{N+1}^0 = \bar{\pi}_{N+1}^0 \\
\pi_A^{N+1} &= -i \bar{\pi}_0^A, \quad \pi_{N+1}^A = i \bar{\pi}_A^0 \\
\pi_B^A + \bar{\pi}_A^B &= 0 \\
\text{tr } \pi &= 0,
\end{aligned}$$

and π satisfies the structure equation

$$-d\pi = \pi \wedge \pi. \quad (3)$$

It is well known that the $\text{SU}(N+1, 1)$ -invariant CR structure on $\Sigma^N \subset \mathbb{C}P^{N+1}$ as a real hypersurface is biholomorphically equivalent to the standard CR structure on $S^{2N+1} = \partial \mathbf{B}^{N+1}$, where $\mathbf{B}^{N+1} \subset \mathbb{C}^{N+1}$ is the unit ball. The structure equation (2) shows that for any local section $s : \Sigma^N \rightarrow \text{SU}(N+1, 1)$, this CR structure is defined by the hyperplane fields $(s^*\pi_0^{N+1})^\perp = \mathcal{H}$ and the set of $(1, 0)$ -forms $\{s^*\pi_0^A\}$.

Definition. Let M be a manifold of dimension $2n+1$. A submanifold defined by an immersion $f : M \hookrightarrow \Sigma^N$ is a *CR submanifold* if $f_*T_pM \cap \mathcal{H}_{f(p)}$ is a complex subspace of $\mathcal{H}_{f(p)}$ of dimension n for each $p \in M$.

Note the induced hyperplane fields $f_*^{-1}(f_*(TM) \cap \mathcal{H})$ is necessarily a contact structure on M , and thus M has an induced nondegenerate CR structure of hypersurface type. When M is equipped with a CR structure, an immersion $f : M \rightarrow \Sigma^N$ is CR when the CR structure induced by f is equivalent to the given one.

Consider $f^*\text{SU}(N+1, 1) \rightarrow M$. From the definition, we may arrange so that $\pi_0^\alpha = 0$ for $n+1 \leq \alpha \leq N = n+m$ on this bundle. Differentiating this, we get

$$\pi_i^\alpha \wedge \pi_0^i + \pi_{N+1}^\alpha \wedge \pi_0^{N+1} = 0.$$

By Cartan's lemma,

$$\pi_i^\alpha \equiv H_{ij}^\alpha \pi_0^j \quad \text{mod } \pi_0^{N+1}, \quad (4)$$

for a coefficient $H_{ij}^\alpha = H_{ji}^\alpha$. H_{ij}^α represents the second fundamental form of f [EHZ].

Our proof of **Theorem** is based on the following algebraic theorem due to Iwatani on the asymptotic subspace of the second fundamental form of a Bochner-Kähler submanifold [Iw][Br]. Let $V = \mathbb{C}^n$, $W = \mathbb{C}^m$ with the standard Hermitian scalar product. Let $\{z^i\}$, $1 \leq i \leq n$, be a unitary (1,0)-basis for V^* , and $\{w_\alpha\}$, $1 \leq \alpha \leq m$, be a unitary basis for W . Let $S^{p,q}$ denote the space of polynomials of type (p, q) on V .

Theorem [Iw] *Suppose $H = H_{ij}^\alpha z^i z^j \otimes w_\alpha \in S^{2,0} \otimes W$ satisfies*

$$\gamma(H, H) = H_{ij}^\alpha \bar{H}_{kl}^\alpha z^i z^j \otimes \bar{z}^k \bar{z}^l = (z^k \bar{z}^k) h \in S^{2,2}, \quad h \in S^{1,1},$$

or simply $\gamma(H, H)$ is Bochner-flat [Br]. Then the asymptotic vectors $\{v \in V \mid H(v, v) = 0\}$ form a subspace of V . Let $n - k$ be the dimension of this asymptotic subspace. Then,

$$m \geq \frac{1}{2} k (2n - k + 1).$$

We define k to be the *rank* of H . Note when $m < n$, $k = 0$, and when $m < 2n - 1$, k is at most 1.

In the case of our interest, the codimension of the CR submanifold is bounded by $N - n = (3n - 2) - n < 2n - 1$, and hence $k \leq 1$. Up to a unitary transformation on V , we may thus arrange

$$H_{ij}^\alpha = H_i^\alpha \delta_{jn} + H_j^\alpha \delta_{in},$$

for coefficients H_i^α . Set $\nu_i = H_i^\alpha w_\alpha \in W$. A computation shows $\gamma(H, H)$ is Bochner-flat when $\langle \nu_i, \nu_j \rangle = 0$ for $i \neq j$, and $\langle \nu_i, \nu_i \rangle = \langle \nu_j, \nu_j \rangle$ for all i, j . Up to a unitary transformation of W , we may set

$$\nu_i = \lambda w_i,$$

for some $\lambda \geq 0$.

Let $f : \Sigma^n \hookrightarrow \Sigma^N$, $N = 3n - 2$, be a local CR immersion. Since Σ^n is CR flat, after identifying $V = f_* T^{\mathbb{C}} \Sigma^n \simeq \mathbb{C}^n \subset \mathcal{H}$ and $W = V^\perp \simeq \mathbb{C}^{N-n} \subset \mathcal{H}$, the second fundamental form of f is Bochner-flat [EHZ].

Suppose $N - n < n$. Then $H \equiv 0$, and f is easily seen to be a part of the linear embedding $f : \Sigma^n \hookrightarrow \Sigma^n \subset \Sigma^N$

Suppose $n \leq N - n < 2n - 1$. Then H has rank at most 1, and from the argument above we have in (4),

$$\begin{aligned} \pi_q^{n+i} &\equiv \lambda \delta_{iq} \pi_0^n \quad \text{mod } \pi_0^{N+1}, \quad \text{for } q < n, i \leq n, \\ \pi_n^{n+i} &\equiv \lambda(1 + \delta_{in}) \pi_0^i \quad \text{mod } \pi_0^{N+1}, \quad \text{for } i \leq n, \\ \pi_i^a &\equiv 0 \quad \text{mod } \pi_0^{N+1}, \quad \text{for } a > 2n, \end{aligned} \tag{5}$$

for a coefficient $\lambda \geq 0$.

2 Proof of theorem

We wish to explore the consequence of the successive derivatives of the relation (5). To facilitate the computation, we shall agree on the index range

$$\begin{aligned} 1 \leq p, q, s, t \leq n - 1, \quad p' = n + p, \\ 1 \leq i, j, k, l \leq n, \quad i' = n + i, \\ n' + 1 \leq a, b \leq m = n' + r. \end{aligned}$$

The induced Maurer-Cartan form (2) on $f^*\text{SU}(N+1,1)$ decomposes according to these indices as follows.

$$\pi = \begin{pmatrix} \pi_0^0 & \pi_q^0 & \pi_n^0 & \pi_{q'}^0 & \pi_{n'}^0 & \pi_b^0 & \pi_{N+1}^0 \\ \pi_0^p & \pi_q^p & \pi_n^p & \pi_{q'}^p & \pi_{n'}^p & \pi_b^p & \pi_{N+1}^p \\ \pi_0^n & \pi_q^n & \pi_n^n & \pi_{q'}^n & \pi_{n'}^n & \pi_b^n & \pi_{N+1}^n \\ \cdot & \pi_q^{p'} & \pi_n^{p'} & \pi_{q'}^{p'} & \pi_{n'}^{p'} & \pi_b^{p'} & \pi_{N+1}^{p'} \\ \cdot & \pi_q^{n'} & \pi_n^{n'} & \pi_{q'}^{n'} & \pi_{n'}^{n'} & \pi_b^{n'} & \pi_{N+1}^{n'} \\ \cdot & \pi_q^a & \pi_n^a & \pi_{q'}^a & \pi_{n'}^a & \pi_b^a & \pi_{N+1}^a \\ \pi_0^{N+1} & \pi_q^{N+1} & \pi_n^{N+1} & \cdot & \cdot & \cdot & \pi_{N+1}^{N+1} \end{pmatrix}, \quad (6)$$

where ' \cdot ' denotes 0. We denote $\pi_0^i = \eta^i$, $\pi_0^{N+1} = \theta$, and $-d\theta \equiv i\eta^k \wedge \bar{\eta}^k = i\varpi \pmod{\theta}$ for the sake of notation. Equation (5) for example can now be written as

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \\ \pi_q^a & \pi_n^a \end{pmatrix} \equiv \begin{pmatrix} \lambda \delta_{pq} \eta^n & \lambda \eta^p \\ \cdot & 2\lambda \eta^n \\ \cdot & \cdot \end{pmatrix} \pmod{\theta}.$$

The case $\lambda \equiv 0$ ($H \equiv 0$) has already been identified with a part of the linear embedding. Suppose $\lambda \neq 0$, and H has rank 1. We may scale $\lambda = 1$ using the group action by $\text{Re } \pi_0^0$, and obtain the following normalized structure equation for a nonlinear local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^N$ with second fundamental form of rank 1,

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \\ \pi_q^a & \pi_n^a \end{pmatrix} = \begin{pmatrix} \delta_{pq} \eta^n & \eta^p \\ \cdot & 2\eta^n \\ \cdot & \cdot \end{pmatrix} + \begin{pmatrix} h_q^{p'} & h_n^{p'} \\ h_q^{n'} & h_n^{n'} \\ h_q^a & h_n^a \end{pmatrix} \theta, \quad (7)$$

for coefficients $h_j^{i'}$, h_j^a .

Theorem is obtained by successive application of Maurer-Cartan equation (3) to this structure equation. We assume $n \geq 4$ for simplicity for the rest of this section, as $n = 3$ case can be treated with a minor modification. The expression "differentiate X mod Y" would mean "differentiate X and considering mod Y".

Step 1. Differentiate $\pi_s^{n'} = h_s^{n'} \theta \pmod{\theta}$, we get

$$\mathbf{i} h_s^n \varpi \equiv (\pi_{s'}^{n'} - 2\pi_s^n) \wedge \eta^n + \pi_{N+1}^{n'} \wedge (-\mathbf{i} \bar{\eta}^s) \pmod{\theta}.$$

Since $n - 1 \geq 2$, this implies $h_s^{n'} = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{n'} - 2\pi_s^n \\ \pi_{N+1}^{n'} \end{pmatrix} \equiv \begin{pmatrix} 2c_s & u \\ u & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ -\mathbf{i} \bar{\eta}^s \end{pmatrix} \pmod{\theta}$$

for coefficients c_s, u .

Differentiate $\pi_s^a = h_s^a \theta \pmod{\theta}$, we get

$$\mathbf{i} h_s^a \varpi \equiv \pi_{s'}^a \wedge \eta^n + \pi_{N+1}^a \wedge (-\mathbf{i} \bar{\eta}^s) \pmod{\theta}.$$

Since $n - 1 \geq 2$, this implies $h_s^a = 0$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^a \\ \pi_{N+1}^a \end{pmatrix} \equiv \begin{pmatrix} 2C_s^a & u^a \\ u^a & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ -\mathbf{i} \bar{\eta}^s \end{pmatrix} \pmod{\theta}$$

for coefficients C_s^a, u^a .

Differentiate $\pi_n^a = h_n^a \theta \pmod{\theta}$, we get

$$\mathbf{i} h_n^a \varpi \equiv \pi_{p'}^a \wedge \eta^p + \pi_{n'}^a \wedge 2\eta^n + \pi_{N+1}^a \wedge (-\mathbf{i} \bar{\eta}^n) \pmod{\theta}.$$

Thus $h_n^a = u^a$, and

$$\pi_{n'}^a \equiv C_p^a \eta^p + C_n^a \eta^n - \mathbf{i} u^a \bar{\eta}^n \pmod{\theta}$$

for coefficients C_n^a .

Step 2. Differentiate $\pi_s^{t'} = h_s^{t'} \theta \pmod{\theta}$ for $t \neq s$, we get

$$\mathbf{i} h_s^{t'} \varpi \equiv (\pi_{s'}^{t'} - \pi_s^t) \wedge \eta^n - \pi_s^n \wedge \eta^t + \pi_{N+1}^{t'} \wedge (-\mathbf{i} \bar{\eta}^s) \pmod{\theta}.$$

For $n - 1 \geq 3$, this implies $h_s^{t'} = 0$ for $t \neq s$, and by Cartan's lemma

$$\begin{pmatrix} \pi_{s'}^{t'} - \pi_s^t \\ -\pi_s^n \\ \pi_{N+1}^{t'} \end{pmatrix} \equiv \begin{pmatrix} 0 & b_s & -\mathbf{i} \bar{b}_t \\ b_s & 0 & e \\ -\mathbf{i} \bar{b}_t & e & 0 \end{pmatrix} \begin{pmatrix} \eta^n \\ \eta^t \\ -\mathbf{i} \bar{\eta}^s \end{pmatrix} \pmod{\theta}$$

for coefficients b_s, e . Since $\pi_{s'}^{t'} - \pi_s^t$ is skew Hermitian, it cannot have any η^n -term.

Step 3. Differentiate $\pi_t^{t'} = \eta^n + h_t^{t'} \theta \pmod{\theta}$, we get

$$h_t^{t'} \varpi \equiv \Delta_t \wedge \eta^n + (b_p \eta^n - i e \bar{\eta}^p) \wedge \eta^p + (-b_t \eta^t + \bar{b}_t \bar{\eta}^t) \wedge \eta^n \pmod{\theta},$$

where $\Delta_t = \pi_{t'}^{t'} - \pi_t^t + \pi_0^0 - \pi_n^n$. This implies $h_t^{t'} = e$, and

$$\Delta_t = a_t \eta^n - i e \bar{\eta}^n + (b_t \eta^t - \bar{b}_t \bar{\eta}^t) + \sum_p b_p \eta^p - A_t \theta,$$

for coefficients a_t, A_t .

Step 4. Since $h_s^{t'} = \delta_{st} e$ and from (7), we may add $e \theta$ to η^n to translate $e = 0$, which we assume from now on. We also translate $h_n^{t'} = 0$ similarly by adding $h_n^{t'} \theta$ to η^t . Differentiating $\pi_n^{t'} = \eta^t \pmod{\theta}$ with these relations and collecting terms, we get $b_t = c_t = 0$, and

$$0 \equiv a_t \eta^n \wedge \eta^t - 2i \bar{u} \eta^t \wedge \eta^n \pmod{\theta}.$$

Thus $a_t = -2i \bar{u}$.

Step 5. Differentiate $\pi_n^{n'} = 2\eta^n + h_n^{n'} \theta \pmod{\theta}$, and collecting terms, we get $h_n^{n'} = u$, and $0 \equiv \Delta_n \wedge \eta^n - iu \eta^n \wedge \bar{\eta}^n \pmod{\theta}$, where $\Delta_n = \pi_{n'}^{n'} - \pi_n^n + \pi_0^0 - \pi_n^n$. Hence

$$\Delta_n = -iu \bar{\eta}^n + a_n \eta^n - A_n \theta,$$

for coefficients a_n, A_n . But $\Delta_t - \Delta_n$ is purely imaginary, and comparing with *Step 3*, $a_n = -3i \bar{u}$.

Step 6. Now by considering θ -terms in *Step 1, 2, 3, 4, 5*, and the fact that $\pi_i^\alpha \wedge \eta^i + \pi_{N+1}^\alpha \wedge \theta = 0$, we obtain the following simple structure equations. We omit the details of computations.

$$\begin{pmatrix} \pi_q^{p'} & \pi_n^{p'} \\ \pi_q^{n'} & \pi_n^{n'} \\ \pi_q^a & \pi_n^a \end{pmatrix} = \begin{pmatrix} \delta_{pq} \eta^n & \eta^p \\ 0 & 2\eta^n + u\theta \\ 0 & u^a \theta \end{pmatrix}$$

$$\begin{pmatrix} \pi_{N+1}^{p'} \\ \pi_{N+1}^{n'} \\ \pi_{N+1}^a \end{pmatrix} = \begin{pmatrix} 0 \\ u\eta^n \\ u^a \eta^n \end{pmatrix}$$

$$\begin{pmatrix} \pi_{s'}^{n'} \\ \pi_{s'}^a \end{pmatrix} = \begin{pmatrix} -iu\bar{\eta}^s \\ -iu^a \bar{\eta}^s + 2C_s^a \eta^n \end{pmatrix}$$

$$\pi_s^n = 0, \pi_{s'}^{t'} = \pi_s^t \text{ for } t \neq s,$$

$$\pi_{n'}^a = C_p^a \eta^p + C_n^a \eta^n - iu^a \bar{\eta}^n + h_{n'}^a \theta,$$

$$\begin{pmatrix} \pi_{N+1}^t \\ \pi_{N+1}^n \end{pmatrix} = \begin{pmatrix} (A - i(u\bar{u} + u^a \bar{u}^a)) \eta^t - 2u^a \bar{C}_t^a \bar{\eta}^n + B_t \theta \\ A \eta^n + B_n \theta \end{pmatrix}$$

$$\begin{pmatrix} \Delta_t \\ \Delta_n \end{pmatrix} = \begin{pmatrix} -2i\bar{u}\eta^n - A\theta \\ -3i\bar{u}\eta^n - iu\bar{\eta}^n - A_n \theta \end{pmatrix}$$

$$A_n + \bar{A}_n = A + \bar{A}$$

$$du = u(\pi_{N+1}^{N+1} - \pi_0^0 + \pi_n^n - \pi_{n'}^{n'}) + 2(A - A_n)\eta^n - u^a \pi_a^{n'} + u_0 \theta$$

$$du^a = u^a(\pi_{N+1}^{N+1} - \pi_0^0 + \pi_n^n) - u^b \pi_b^a - u \pi_{n'}^a + 2h_{n'}^a \eta^n + u_0^a \theta \quad (8)$$

Proof of Theorem. From [HJX, Corollary 1.4], a proper holomorphic map $F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{3n-1}$ which is C^3 up to the boundary is rational, in particular real analytic up to the boundary. Hamada recently showed that a rational proper holomorphic map from \mathbf{B}^{n+1} to \mathbf{B}^{2n+2} , $n \geq 3$, is equivalent to one of the three class of maps in **Theorem**. It thus suffices to show that any analytic local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ in fact lies in $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$ for $n \geq 3$.

Step 1. From the refined structure equations above, differentiate $\pi_{N+1}^{p'} = 0$, $\pi_{N+1}^{n'} =$

$u \eta^n$, and $\pi_{N+1}^a = u^a \eta^n$. After a short computation, we get $B_p = 0$, $B_n = 0$, and

$$u_0 = -2uA, \quad u_0^a = -2u^a A. \quad (9)$$

Differentiating $\pi_q^n = 0$, and $\pi_{s'}^{t'} = \pi_s^t$ for $t \neq s$ with these relations, we get

$$\begin{aligned} A - \bar{A} &= i(u\bar{u} + u^a\bar{u}^a - 1), \\ \sum_{a=n'+1}^{n'+r} C_t^a \bar{C}_s^a &= 0, \quad \text{for } t \neq s. \end{aligned} \quad (10)$$

Step 2. Differentiate $i\pi_{q'}^{n'} = u\bar{\eta}^q$, and collecting $\eta^n \wedge \bar{\eta}^q$ -terms, we get

$$\sum_{a=n'+1}^{n'+r} C_q^a \bar{C}_q^a = 1 - u\bar{u} + i(A - A_n),$$

which is independent of the index q . Since $r < n - 1$ from our assumption on the codimension, this and (10) force

$$C_q^a = 0, \quad (11)$$

and $A_n = A + i(u\bar{u} - 1)$.

Step 3. Differentiate $\pi_{n'}^a = C_n^a \eta^n - iu^a \bar{\eta}^n + h_{n'}^a \theta \pmod{\theta, \eta^n, \bar{\eta}^n}$, we get

$$h_{n'}^a = -iu^a \bar{u}. \quad (12)$$

Step 4. Differentiate $\pi_{N+1}^p = (\bar{A} - i)\eta^p$, $\pi_{N+1}^n = A\eta^n$, we get

$$dA = A(\pi_{N+1}^{N+1} - \pi_0^0) + \pi_{N+1}^0 + 2(u\bar{\eta}^n - \bar{u}\eta^n) + (u\bar{u} + u^a\bar{u}^a - A^2)\theta.$$

Note $\Delta_t + \bar{\Delta}_t = \pi_0^0 + \bar{\pi}_0^0 = \pi_0^0 - \pi_{N+1}^{N+1} = 2i(u\bar{\eta}^n - \bar{u}\eta^n) - (A + \bar{A})\theta$. Now finally differentiating $\Delta_n = -3i\bar{u}\eta^n - iu\bar{\eta}^n - A_n\theta$ with these relations, and collecting $\eta^n \wedge \bar{\eta}^n$ -terms, we get $\sum_{a=n'+1}^{n'+r} C_n^a \bar{C}_n^a = 0$, and hence

$$C_n^a = 0. \quad (13)$$

Case B. Suppose $u^a = 0$ for all a . From (11), (12), (13), $\pi_n^a = \pi_{s'}^a = \pi_{n'}^a = \pi_{N+1}^a = 0$. In the notation of (2), the complex $(2n+2)$ -plane $Z_0 \wedge Z_1 \wedge \dots \wedge Z_{2n} \wedge Z_{N+1}$ is then constant along the CR immersion f . Hence $f : \Sigma^n \hookrightarrow \Sigma^{2n} \subset \Sigma^{3n-2}$.

Case **C**. Suppose $\vec{u} = (u^{n'+1}, \dots, u^{n'+r}) \neq 0$. Using a group action by π_b^a , we may rotate so that $\vec{u} = (u^{n'+1}, 0, \dots, 0)$ with $u^{n'+1} \neq 0$. But from (11), (12), (13), we have $\pi_n^a = \pi_{s'}^a = \pi_{n'}^a = \pi_{N+1}^a = 0$ for $a > n' + 1$. Moreover, (8) shows $\pi_{n'+1}^a = 0$ for $a > n' + 1$. In the notation of (2), the complex $(2n + 3)$ -plane $Z_0 \wedge Z_1 \wedge \dots \wedge Z_{2n} \wedge Z_{2n+1} \wedge Z_{N+1}$ is then constant along the CR immersion f . Hence $f : \Sigma^n \hookrightarrow \Sigma^{2n+1} \subset \Sigma^{3n-2}$. \square

At this stage, note that the only possibly independent coefficients in the structure equations are A, u, u^a , and that the expression for their derivatives does not involve any new variables. The structure equations for local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{3n-2}$ thus close up at order 3. A long but direct computation shows that these equations are compatible, i.e., $d^2 = 0$ is a formal identity of the structure equation. In [Wa], we showed that every local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n}$ is equivalent to a part of either the linear embedding or Whitney map. We suspect a similar argument can be applied to show that every linearly full local CR immersion $f : \Sigma^n \hookrightarrow \Sigma^{2n+1}$ is equivalent to a boundary CR map of a type **C** proper holomorphic map in **Theorem**.

Remark. The computation involved here is reminiscent of Cartan's local isometric embedding of Hyperbolic space \mathbb{H}^n in Euclidean space \mathbb{E}^{2n-1} via exteriorly orthogonal symmetric bilinear forms [Ca]. Overdetermined nature of CR geometry forces the structure equation to close up instead of becoming involutive.

Remark. It is natural to ask the Kähler analogue of this rigidity theorem. Let $M^n \hookrightarrow X_\epsilon^N$ be a complex submanifold in a complex space form X of constant holomorphic sectional curvature ϵ . Suppose the induced metric on M is Bochner-Kähler and the codimension is bounded by $N - n < 2n - 1$. A short computation shows that such M is totally geodesic for $n \geq 3$.

The computation suggests that the gap phenomena in **Theorem** may persist for CR immersions between spheres with second fundamental form of rank 1; *linearly full CR immersions $f : \Sigma^n \hookrightarrow \Sigma^N$, $N \leq n^2 - 2$, with second fundamental form of rank 1 can only occur when $N = \mu n, \mu n + 1, \dots, \mu n + \mu - 1$ for $\mu \leq n - 1$.*

Consider the following generalization of Whitney map.

$$F : \mathbf{B}^{n+1} \rightarrow \mathbf{B}^{\mu(n+1)}, \quad (14)$$

$$F(z^i, z^0) = (x_1 z^i, x_2 z^i z^0, \dots, x_\mu z^i (z^0)^{\mu-1}, y_1 z^0, y_2 z^0 z^0, \dots, y_\mu z^0 (z^0)^{\mu-1}).$$

where x_A, y_A are constants satisfying $x_1 = 1, x_\mu = y_\mu, x_A^2 = y_A^2 + x_{A+1}^2$ for $1 \leq A \leq \mu-1$. The geometric rank of a proper holomorphic map between unit balls introduced by Huang is likely bounded above by the rank of the second fundamental form of its boundary CR map at a generic point [Hu]. It would be interesting to understand how exhaustive (14) is among the set of proper holomorphic maps with geometric rank 1 from \mathbf{B}^{n+1} to $\mathbf{B}^{\mu(n+1)}$ for $\mu \leq n-1$.

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